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ON A CLASS OF STARLIKE FUNCTIONS RELATED WITH BOOTH LEMNISCATE

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Abstract. Let $\mathcal{A}$ be the class of normalized analytic functions. We study the class $\mathcal{BS}(\alpha)$ as follows

$$\mathcal{BS}(\alpha) := \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{z}{1-\alpha z^2},\ |z|<1 \right\},$$

where "$\prec$" is the subordination relation and $0 \le \alpha < 1$. Also, we consider the class $\mathcal{BK}(\alpha)$, including of all functions $f \in \mathcal{A}$ such that $zf'(z) \in \mathcal{BS}(\alpha)$. In the present article, some properties of the classes $\mathcal{BS}(\alpha)$ and $\mathcal{BK}(\alpha)$ are investigated.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ including of functions normalized by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$ consists of all univalent functions $f(z)$ in Δ is denoted by $\mathcal{S}$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in Δ will be denoted by $\mathcal{S}^*$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in Δ by $\mathcal{K}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E.

We now recall from [9, 10], a one-parameter family of functions as follows

$$F_\alpha(z) := \frac{z}{1-\alpha z^2} = \sum_{n=1}^{\infty} \alpha^{n-1} z^{2n-1} \qquad (z \in \Delta,\ 0 \le \alpha \le 1), \tag{1.1}$$

where $F_\alpha(z)$ is a starlike univalent function when $0 \le \alpha < 1$.

Now we recall the following lemma.

Lemma 1.1. *(see [3]) Let $F_\alpha(z)$ be given by (1.1). Then for $0 \le \alpha < 1$, we have*

$$\frac{1}{\alpha - 1} < \mathfrak{Re}\{F_\alpha(z)\} < \frac{1}{1-\alpha} \qquad (z \in \Delta). \tag{1.2}$$

We have also $\partial F_\alpha(\Delta) = D(\alpha)$, where

$$D(\alpha) = \left\{ x+iy \in \mathbb{C} :\ \left(x^2+y^2\right)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} = 0 \right\}, \tag{1.3}$$

when $0 \le \alpha < 1$ and

$$D(1) = \{x+iy \in \mathbb{C} :\ (\forall t \in (-\infty, -i/2] \cup [i/2, \infty))\, [x+iy \ne it]\}. \tag{1.4}$$

We remark that a curve described by

$$\left(x^2+y^2\right)^2 - \left(n^4+2m^2\right)x^2 - \left(n^4-2m^2\right)y^2 = 0 \qquad (x,y) \ne (0,0), \tag{1.5}$$

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is called the Booth lemniscate, named after J. Booth [1]. For more details see [3]. Robertson introduced in [12], the class $\mathcal{S}^*(\gamma)$ of starlike functions of order $\gamma \le 1$, which is defined by

$$\mathcal{S}^*(\gamma) := \left\{ f \in \mathcal{A} : \ \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma,\ z \in \Delta \right\}.$$

If $\gamma \in [0,1)$, then a function in $\mathcal{S}^*(\gamma)$ is univalent. Also, the function $f \in \mathcal{K}(\gamma)$, if and only if $zf'(z) \in \mathcal{S}^*(\gamma)$. In particular we put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$. By $\mathcal{Q}^*(\delta)$, $\delta > 1$, we understand the class of functions

$$\mathcal{Q}^*(\delta) := \left\{ f \in \mathcal{A} : \ \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \delta,\ z \in \Delta \right\}.$$

We denote by $\mathfrak{B}$ the class of analytic functions $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \Delta)$. If f and g are two of the functions in $\mathcal{A}$, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a $w \in \mathfrak{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$.

Furthermore, if the function g is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

We now recall from [3] the following definition.

Definition 1.1. *The function $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$, $0 \le \alpha < 1$, if it satisfies the condition*

$$\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{z}{1 - \alpha z^2} \qquad (z \in \Delta). \tag{1.6}$$

By Lemma 1.1, if $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$, then

$$\frac{\alpha}{\alpha - 1} < \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \frac{2-\alpha}{1-\alpha} \qquad (z \in \Delta), \tag{1.7}$$

or $f \in \mathcal{S}^*(\frac{\alpha}{\alpha-1}) \cap \mathcal{Q}^*(\frac{2-\alpha}{1-\alpha})$. It is clear that $\mathcal{BS}(0) \subset \mathcal{S}^*$.

The organization of this paper is the following. In the Section 2 we obtain some further results about the class $\mathcal{BS}(\alpha)$. Also, in the Section 3, we consider by $\mathcal{BK}(\alpha)$ a subclass of univalent functions connected with $\mathcal{BS}(\alpha)$ and we obtain some properties of it.

2. The class $\mathcal{BS}(\alpha)$

To proves our main results we shall need the following lemma.

Lemma 2.1. *A function f belongs to the class $\mathcal{BS}(\alpha)$ if and only if there exists an analytic function q, $q(0) = 0$ and $q \prec F_\alpha$ such that*

$$f(z) = z \exp\left(\int_0^z \frac{q(t)}{t} \mathrm{d}t \right). \tag{2.1}$$

Proof. Assume that $f \in \mathcal{BS}(\alpha)$ and let $q(z) = [zf'(z)/f(z)] - 1$. Then $q(z) \prec F_\alpha(z)$ and integrating this equation we obtain (2.1). If f is given by (2.1), with an analytic q, $q(0) = 0$, $q \prec F_\alpha$, then differentiating logarithmically (2.1) we obtain $[zf'(z)/f(z)] - 1 = q(z)$ therefore $[zf'(z)/f(z)] - 1 \prec F_\alpha(z)$ and $f \in \mathcal{BS}(\alpha)$. □

If we apply Lemma 2.1 with $q(z) = F_\alpha(z)$, then (2.1) becomes

$$\begin{aligned} f_\alpha(z) &:= z\exp\left(\int_0^z \frac{F_\alpha(t)}{t}\mathrm{d}t\right) \\ &= z\exp\left(\int_0^z \frac{\mathrm{d}t}{1-\alpha t^2}\right) \\ &= z\exp\left(\int_0^z \frac{1}{2}\left(\frac{t}{1-\sqrt{\alpha}t}+\frac{t}{1+\sqrt{\alpha}t}\right)\mathrm{d}t\right) \\ &= z\exp\left(\frac{1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right) \\ &= z\left(\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right)^{1/(2\sqrt{\alpha})} \end{aligned}$$

and so $f_\alpha(z) \in \mathcal{BS}(\alpha)$. Moreover, we have

$$f_\alpha(z) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{3}\left(\alpha+\frac{1}{2}\right)z^4 + \frac{1}{12}\left(4\alpha+\frac{1}{2}\right)z^5 + \cdots. \tag{2.2}$$

The function $f_\alpha(z)$ is extremal function for several problems in the class $\mathcal{BS}(\alpha)$.

Theorem 2.1. *If a function $f \in \mathcal{BS}(\alpha)$, then there exists a function $G \in \mathcal{S}^*(\sqrt{\alpha}/(1+\sqrt{\alpha}))$ and a function $H \in \mathcal{Q}^*((2+\sqrt{\alpha})/(1+\sqrt{\alpha}))$ such that*

$$\frac{f(z)}{z} = \left[\frac{G(z)}{z}\right]^{\frac{1}{2}} \times \left[\frac{H(z)}{z}\right]^{\frac{1}{2}} \qquad (z \in \Delta). \tag{2.3}$$

Proof. Let $f \in \mathcal{BS}(\alpha)$. Then by Lemma 2.1,

$$f(z) = z\exp\left(\int_0^z \frac{F_\alpha(w(t))}{t}\mathrm{d}t\right), \tag{2.4}$$

where $|w(z)| \le |z|$. Easily seen that

$$F_\alpha(z) = \frac{1}{2}\left(\frac{z}{1-\sqrt{\alpha}z}+\frac{z}{1+\sqrt{\alpha}z}\right).$$

Hence, we can rewrite (2.4) in the form

$$\begin{aligned} f(z) &= z\exp\left[\frac{1}{2}\left(\int_0^z \frac{w(t)}{t(1-\sqrt{\alpha}w(t))}\mathrm{d}t + \int_0^z \frac{w(t)}{t(1+\sqrt{\alpha}w(t))}\mathrm{d}t\right)\right] \\ &= z\left[\exp\left(\int_0^z \frac{w(t)}{t(1-\sqrt{\alpha}w(t))}\mathrm{d}t\right)\right]^{\frac{1}{2}} \times \left[\exp\left(\int_0^z \frac{w(t)}{t(1+\sqrt{\alpha}w(t))}\mathrm{d}t\right)\right]^{\frac{1}{2}} \\ &= z\left[\frac{G(z)}{z}\right]^{\frac{1}{2}} \times \left[\frac{H(z)}{z}\right]^{\frac{1}{2}} \qquad (2.5) \end{aligned}$$

where

$$\frac{G(z)}{z} = \exp\left(\int_0^z \frac{w(t)}{t(1-\sqrt{\alpha}w(t))}\mathrm{d}t\right), \tag{2.6}$$

and

$$\frac{H(z)}{z} = \exp\left(\int_0^z \frac{w(t)}{t(1+\sqrt{\alpha}w(t))}\mathrm{d}t\right). \tag{2.7}$$

From (2.6), we have

$$\frac{zG'(z)}{G(z)} = \frac{1+(1-\sqrt{\alpha})w(z)}{1-\sqrt{\alpha}w(z)}.$$

Since

$$\mathrm{Re}\left\{\frac{1+(1-\sqrt{\alpha})z}{1-\sqrt{\alpha}z}\right\} > \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}},$$

hence we have $G(z) \in \mathcal{S}^*(\sqrt{\alpha}/(1+\sqrt{\alpha}))$. On the other hand, we have

$$\frac{zH'(z)}{H(z)} = \frac{1+(1+\sqrt{\alpha})w(z)}{1+\sqrt{\alpha}w(z)},$$

which shows that $H(z) \in \mathcal{Q}^*((2+\sqrt{\alpha})/(1+\sqrt{\alpha}))$. This completes the proof. □

Theorem 2.2. *Assume that* $0 \le \alpha < 1$. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{BS}(\alpha)$, *then*

$$\sum_{n=2}^{\infty} \left(n^2 (1-\alpha)^2 - (2+\alpha)^2\right) |a_n|^2 \le 3 - 2\alpha. \tag{2.8}$$

Proof. Let $f \in \mathcal{BS}(\alpha)$. The function F_α is analytic in the unit disc, so to find the image of the circle $|z| = 1$ under the function F_α consider

$$\begin{aligned} F_\alpha(e^{i\varphi}) &= \frac{e^{i\varphi}}{1-\alpha e^{2i\varphi}} \cdot \frac{1-\alpha e^{-2i\varphi}}{1-\alpha e^{-2i\varphi}} = \frac{e^{i\varphi} - \alpha e^{-i\varphi}}{1+\alpha^2 - \alpha(e^{2i\varphi}+e^{-2i\varphi})} \\ &= \frac{(1-\alpha)\cos\varphi + i(1+\alpha)\sin\varphi}{1+\alpha^2 - 2\alpha\cos 2\varphi}. \end{aligned}$$

If

$$x = \mathfrak{Re}\left\{F_\alpha(e^{i\varphi})\right\} = \frac{(1-\alpha)\cos\varphi}{1+\alpha^2-2\alpha\cos 2\varphi}, \tag{2.9}$$

and

$$y = \mathfrak{Im}\left\{F_\alpha(e^{i\varphi})\right\} = \frac{(1+\alpha)\sin\varphi}{1+\alpha^2-2\alpha\cos 2\varphi}, \tag{2.10}$$

then after some calculation we can obtain from (2.9) and (2.10) that

$$x^2 + y^2 = \frac{1}{1+\alpha^2-2\alpha\cos 2\varphi}.$$

Therefore, using this we can find that

$$\begin{aligned} |F_\alpha(e^{i\varphi})|^2 &= \left|\frac{1}{1+\alpha^2-2\alpha\cos 2\varphi}\right|^2 \\ &\le \left|\frac{1}{1+\alpha^2-2\alpha}\right|^2 \\ &= \left(\frac{1}{1-\alpha}\right)^2, \end{aligned}$$

and so

$$|F_\alpha(z)| < \frac{1}{1-\alpha} \qquad (z \in \Delta). \tag{2.11}$$

From (1.6) and (2.11), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{1}{1-\alpha},$$

which gives

$$\left|\frac{zf'(z)}{f(z)}\right| \le \frac{2-\alpha}{1-\alpha},$$

therefore, we get

$$|f(z)|^2 \ge |zf'(z)|^2 \left(\frac{1-\alpha}{2-\alpha}\right)^2. \tag{2.12}$$

Integrating (2.12) around $z = re^{i\theta}$, $\theta \in [0, 2\pi)$ gives

$$\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \geq \left(\frac{1-\alpha}{2-\alpha}\right)^2 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n} \qquad r \in (0,1),$$

If $r \to 1^-$, then it becomes

$$\sum_{n=1}^{\infty} |a_n|^2 \geq \left(\frac{1-\alpha}{2-\alpha}\right)^2 \sum_{n=1}^{\infty} n^2 |a_n|^2.$$

This gives

$$\sum_{n=1}^{\infty} \left(n^2 (1-\alpha)^2 - (2-\alpha)^2\right) |a_n|^2 \leq 0,$$

or

$$\sum_{n=2}^{\infty} \left(n^2 (1-\alpha)^2 - (2-\alpha)^2\right) |a_n|^2 \leq 3 - 2\alpha, \tag{2.13}$$

where $a_1 = 1$. So the proof is completed. □

Putting $\alpha = 0$ in the Theorem 2.2, we have:

Corollary 2.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies*

$$0 < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 2 \qquad (z \in \Delta).$$

Then we have

$$\sum_{n=2}^{\infty} (n^2 - 4)|a_n|^2 \leq 3.$$

Corollary 2.2. *If $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ satisfies*

$$0 < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 2 \qquad (z \in \Delta),$$

then we have

$$|a_n| \leq \sqrt{\frac{3}{n^2 - 4}} \qquad (n = 3, 4, \ldots).$$

Theorem 2.3. *If a function $f \in \mathcal{BS}(\alpha)$ and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then

$$|a_2| \leq 1, \quad |a_3| \leq 1/2,$$

the equality occurs for $f_\alpha(z)$ given in (2.2).

Proof. From (1.6), we have

$$\left(\frac{zf'(z)}{f(z)} - 1\right) = \frac{w(z)}{1 - \alpha w^2(z)} \qquad (z \in \Delta), \tag{2.14}$$

where $w(z) \in \mathcal{B}$. Assume that $w(z) = c_1 z + c_2 z^2 + \cdots$. Comparing the first two coefficients in (2.14) gives

$$a_2 = c_1, \quad 2a_3 = c_2 + c_1 a_2 = c_2 + c_1^2. \tag{2.15}$$

It is known that for $w(z) \in \mathcal{B}$, we have

$$|c_1| \leq 1, \quad |c_2| + |c_1|^2 \leq 1.$$

Using this in (2.15) gives

$$|a_2| \leq 1, \quad 2|a_3| \leq |c_2| + |c_1|^2 \leq 1,$$

and it ends the proof. □

3. The class $\mathcal{BK}(\alpha)$

First, we recall the definition of $\mathcal{BK}(\alpha)$.

Definition 3.1. *Let $0 \leq \alpha < 1$. Then the function $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$ if f satisfies the following subordination:*

$$\frac{zf''(z)}{f'(z)} \prec F_\alpha(z) \qquad (z \in \Delta). \tag{3.1}$$

The function class $\mathcal{BK}(\alpha)$ was introduced by Najmadi *et al.* [5]. The function

$$\hat{f}_\alpha(z) := z + \frac{z^2}{2} + \frac{1}{6}z^3 + \frac{1}{12}\left(\alpha + \frac{1}{2}\right)z^4 + \frac{1}{60}\left(4\alpha + \frac{1}{2}\right)z^5 + \cdots. \tag{3.2}$$

is extremal function for several problems in the class $\mathcal{BK}(\alpha)$. By Alexander's lemma, $f \in \mathcal{BK}(\alpha)$ if and only if $zf' \in \mathcal{BS}(\alpha)$.

Let $\mathcal{LU}$ denote the subclass of $\mathcal{H}$ consisting of all locally univalent functions, namely,

$$\mathcal{LU} = \{f \in \mathcal{H} : f'(0) \neq 0, \quad z \in \Delta\}.$$

We note that $\mathcal{LU}$ is a vector space over $\mathbb{C}$. In the sequel, we recall the definition of norm of a function $f \in \mathcal{LU}$. For $f \in \mathcal{LU}$, we introduce

$$||f|| = \sup_{z\in\Delta}\left(1 - |z|^2\right)\left|\frac{f''(z)}{f'(z)}\right|, \tag{3.3}$$

where the quantity f''/f' is often referred to as pre-Schwarzian derivative of f. It is known that $||f|| < \infty$ if and only if f is uniformly locally univalent, that is, there exists a constant ρ with $\rho(f) > 0$ such that f is univalent in each disk of hyperbolic radius r in Δ. As a consequences of [6, Proposition 1.2],
One can easily see that $||f|| \leq 6$ for any univalent function f in Δ and the equality is attained for the Koebe function $z/(1-z)^2$ and its rotation. Conversely, f is univalent in Δ if $||f|| \leq 1$ and the bound is sharp.

Many authors, have given norm estimates for classical subclass of univalent functions (for example, see [7] and their references). We need the following lemma, which is a reformulated of Ma and Minda [4, Theorem 1].

Lemma 3.1. *Let $\psi \in \mathcal{H}_1 := \{f \in \mathcal{H} : f(0) = 1\}$ be starlike and suppose that $g \in \mathcal{A}$ satisfies the equation*

$$1 + \frac{zg''(z)}{g'(z)} = \psi(z) \qquad (z \in \Delta).$$

Then for $f \in \mathcal{A}$, the condition $1 + zf''(z)/f'(z) \prec \psi(z)$ implies that $f'(z) \prec g'(z)$.

Next, by considering the norm of a locally univalent function given by (3.3), we find sharp norm estimates for functions in $\mathcal{BK}(\alpha)$.

Theorem 3.1. *Let $0 \leq \alpha < 1$ and $f \in \mathcal{BK}(\alpha)$. Then $||f|| \leq 1$. The result is sharp.*

Proof. Let $0 \leq \alpha < 1$. If $f \in \mathcal{BK}(\alpha)$, then from the definition of $\mathcal{BK}(\alpha)$ we have

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{z}{1 - \alpha z^2} =: P_\alpha(z) \qquad (z \in \Delta),$$

where $P_\alpha(z)$ is a starlike function with respect to 1. Assume that $g(z) \in \mathcal{A}$ and satisfies the following equality

$$1+\frac{zg''(z)}{g'(z)}=1+\frac{z}{1-\alpha z^2} \qquad (z\in\Delta).$$

After a simple calculation, we get

$$g'(z)=\exp\left(\frac{1}{\sqrt{\alpha}}\arctan h(\sqrt{\alpha}z)\right).$$

Thus

(3.4)

$$\begin{aligned} g(z) &= \int_0^z g'(t)\mathrm{d}t \\ &= \int_0^z \exp\left(\frac{1}{\sqrt{\alpha}}\arctan h(\sqrt{\alpha}t)\right)\mathrm{d}t \\ &= \frac{1}{\sqrt{\alpha}(2\sqrt{\alpha}+1)}\exp\left(\frac{1}{\sqrt{\alpha}}\arctan h(\sqrt{\alpha}z)\right) \\ &\times \left((2\sqrt{\alpha}+1)\left(F\left(1,1/2\sqrt{\alpha};1+1/2\sqrt{\alpha};-\exp\left(2\arctan h(\sqrt{\alpha}z)\right)+\sqrt{\alpha}z\right)\right)\right) \\ &- \exp\left(2\arctan h(\sqrt{\alpha}z)\right)F\left(1,1+1/2\sqrt{\alpha};2+1/2\sqrt{\alpha};-\exp\left(2\arctan h(\sqrt{\alpha}z)\right)\right), \end{aligned}$$

where F is the Gauss hypergeometric function. Now, the Lemma 3.1 implies that

$$f'(z)\prec g'(z)=\exp\left(\frac{1}{\sqrt{\alpha}}\arctan h(\sqrt{\alpha}z)\right).$$

By definition of subordination, there exists a Schwarz function $w(z)$ so that

$$f'(z)=\exp\left(\frac{1}{\sqrt{\alpha}}\arctan h(\sqrt{\alpha}w(z))\right).$$

Also, by Schwarz-Pick lemma we get

$$|w'(z)|\le\frac{1-|w(z)|^2}{1-|z|^2}.$$

Hence we have

$$\begin{aligned} \left|\frac{f''(z)}{f'(z)}\right| &= \left|\frac{w'(z)}{1-\alpha w^2(z)}\right| \\ &\le \frac{1}{1-\alpha|w(z)|^2}\frac{1-|w(z)|^2}{1-|z|^2}, \end{aligned}$$

which gives that

$$\begin{aligned} ||f|| &\le \sup_{z\in\Delta}\frac{1-|w(z)|^2}{1-\alpha|w(z)|^2} \\ &\le 1. \end{aligned}$$

The equality holds for the function $g(z)\in\mathcal{BK}(\alpha)$ defined in (3.4), i.e. $||g||=1$ and concluding the proof. □

As a consequence of [6], we get the following.

Corollary 3.1. *Let the function f belongs to the class $\mathcal{BK}(\alpha)$. Then f is a univalent function. That means that $\mathcal{BK}(\alpha)\subset\mathcal{S}$.*

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